MATH 215: LECTURE 5- RELATIONS

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Relation is a wide class of important mathematical objects such as functions, orders and and equivalence relation.

1. General relations

In many situation we would like to describe that certain objects relate to other object. To turn relations into a formal mathematical object, we need to define them as sets. First, how would we code that an object a relates to an object b? we can use the ordered pair $\langle a, b\rangle$. A relations describes many such collections, hence it is a set of ordered pairs:

Definition 1.1. A relation from the set A to the set B is set $R \subseteq A \times B$.

Example 1.2. (1) $R = \{\langle 1,2\rangle, \langle 1,3\rangle\}$ is a relation from $\{1,2\}$ tp $\{1,2,3\}$ since

$$R \subseteq \{1,2\} \times \{1,2,3\}$$

. R is also a relation from $\mathbb{R}$ to $\mathbb{N}$.

(2) $\{\langle 1, \sqrt{2}\rangle, \langle 2,4\rangle\}$ is not a relation from $\mathbb{N}$ to $\mathbb{N}$.

(3)

$$id_{\mathbb{N}} = \{\langle n,n\rangle \mid n \in \mathbb{N}\}$$

$$\leq_{\mathbb{N}} = \{\langle n,m\rangle \in \mathbb{N}^2 \mid \exists k \in \mathbb{N}. n+k = m\}, \ <_{\mathbb{N}} = \{\langle n,m\rangle \in \mathbb{N}^2 \mid \exists k \in \mathbb{N}_+. n+k = m\}$$

are three relations from $\mathbb{N}$ to $\mathbb{N}$. Note that

$$\leq = < \cup id_{\mathbb{N}}$$

(4) $A = \{\langle x,y\rangle \in \mathbb{R}^2 \mid x - y \in \mathbb{Q}\}$ for example $\langle 3+\sqrt{2}, \sqrt{2}\rangle \in A$, $\langle 1, \pi\rangle \notin A$.

(5) $R = \{\langle X,Y\rangle \in P(\mathbb{N}) \times P(\mathbb{Z}) \mid X \subseteq Y\}$. R is a relation from $P(\mathbb{N})$ to $P(\mathbb{Z})$.

(6) It is sometimes convinient to imagine a relation as two potato's representing the sets A and B, and then and arrows from A to B. For example, if $R = \{\langle 1,2\rangle, \langle 2,a\rangle, \langle 2,b\rangle\}$ From $\{1,2,3\}$, to $\{2,a,b\}$:

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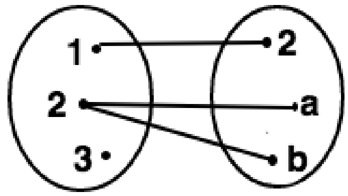

(7) $S = \{\langle x, y\rangle \in \mathbb{Z}^2 \mid x \text{ divides } y\}$, Then S is a relation from $\mathbb{Z}$ to $\mathbb{Z}$.
(8) In general, for every set A we denote the identity relation on the set A by $id_A = \{\langle a, a\rangle \mid a \in A\}$.
(9) A funciton is also a relation. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2$. This function establishes connections between the real number x and the real number x^2, So the formal definition of the function as a set is $f = \{\langle x, x^2\rangle \mid x \in \mathbb{R}\}$.

Remark 1.3. In most cases a relation (i.e. a set of pairs) has a "meaning", which is some notion we already familiar with, just not in terms of sets of pairs. In the previous examples, $\leq_{\mathbb{N}}$ is just a formal representation for the usual $\leq$ where we only consider natural numbers. The relation D is just the divisibility relation on between integers, and id_A is just the equality relation where we only consider elements of the set A. **However**, a general relation R, is just an abstract object. It does not necessarily have a meaning as in the previous examples. Examples $(1), (2), (6)$ do not arise from a natural notion. We can always artificially force a meaning to it, but this would be of no use.

Important: When handling general relations, do not try to find a "meaning" for it. Instead, you should simply think of a set of pairs. When handling a specific relation, it is important to understand the idea behind it (by finding examples pairs of elements which belongs to the relation).

2. Relations on a single set

The first kind of relations we are interested in are relations R from a set A to itself.

Definition 2.1. A relation R from A to A (i.e. $R \subseteq A^2$) is called a relation *on the set A*.

For example, $\leq_{\mathbb{N}}$ is a relation of $\mathbb{N}$, id_A is a relation on A and the divisibility relation S is a relation in $\mathbb{Z}$.

Example 2.2. Let us denote by $\subseteq_A = \{\langle X, Y\rangle \in P(A)^2 \mid X \subseteq Y\}$. Then $\subseteq_A$ is a relation on A.

Instead of writing for example $\langle 2,3\rangle \in \leq_{\mathbb{N}}$ or $\langle\{1\},\{39,1,14\}\rangle \in \subseteq_{\mathbb{Z}}$, we would like to keep the usual notation that $2 \leq_{\mathbb{N}3}$ and $\{1\} \subseteq_{\mathbb{Z}} \{39,1,14\}$. Hence we have the following notation:

Notation 2.3. Given a general relation R on a set A, we define $aRb \equiv \langle a,b\rangle \in R$.

This notation is not convenient when we have relations from a to b. We will have a different notation specifically for functions in the next chapter.

In order to develop some theory and prove interesting theorems about relations, we will need to add more structure/properties to the relation. The most important kind of relations on a single set are *equivalence relations* and *orders*. In this chapter we will only discuss Equivalence realtions.

2.1. **equivalence relations.** As we have seen previously, sets are equal if and only if they have the same elements. This is a quit rigid equality. There are mathematical theories where it is convenient to identify between two objects although they are not equal as sets, we say that they are *equivalent*. For example, to define a rational numbers $\frac{n}{m}$ from the integers, it is natural to identify it with the pair $\langle n,m\rangle$. However, note that while $\frac{1}{2}=\frac{2}{4}$, the pairs $\langle 1,2\rangle,\langle 2,4\rangle$ are distinct. What we usually do, is to set some criterion to determine when two objects are equivalent. Formally, this would mean that we have some relation R on a set A, and two members $a,b\in A$ will be equivalent if aRb. In our example of rationals, we would need to find a criterion which makes $\langle 1,2\rangle,\langle 2,4\rangle$ equivalent for examples, and not only them, but also $\langle 4,2\rangle,\langle 8,2\rangle$ and $\langle -1,9\rangle,\langle 2,-18\rangle$ and so on.

Example 2.4. To find the right criteria for the rations, we need to express the equality $\frac{a}{b}=\frac{c}{d}$ in terms of integers, so let simply cross-multiply the equation and get $ad=bc$. Going back to the beginning, we define a relation R on the **set of pairs** $\mathbb{Z}\times\mathbb{Z}\setminus\{0\}$. Note that this is not a relation on $\mathbb{Z}$, rather then on pairs, and we exclude 0 by only considering pairs of the form $\langle a,b\rangle$ where $b\neq 0$. Now we set the criterion that $\langle a,b\rangle R\langle c,d\rangle$ (namely, the pairs $\langle a,b\rangle$ and $\langle c,d\rangle$ are equivalent) if and only if $ad=bc$. Formally, we define the relation R as follows:

$$R=\Big\{\langle\langle a,b\rangle,\langle c,d\rangle\rangle \in (\mathbb{Z}\times\mathbb{Z}\setminus\{0\})^2 \mid ad=bc\Big\}$$

Since equivalence relations imitate equality, there are some necessary properties which must be posed on a general relation in order for it to be an equivalence relation:

Definition 2.5 (Properties of relations and equivalence relation)**.** Let R be a relation on a set A. We say that:

(1) R is *reflexive* (on A) if: $\forall a\in A.aRa$.
(2) R is *symmetric* if: $\forall a,b\in A.aRb\Rightarrow bRa$.
(3) R is *transitive* if: $\forall a,b,c\in A.(aRb)\wedge(bRc)\Rightarrow aRc$.
(4) R is an *equivalence relation* if it is reflexive, symmetric and transitive.

Example 2.6. (1) Let us give some non mathematical relations on the "set" of all humans to illustrate these properties:

(a) The *brotherhood relation*: two humans x, y are brothers if and only if they have the same biological parents.[1]
The brotherhood relation is reflexive: Indeed, **every** human x is a brother of himself, as by this definition x has the same two biological parents as himself.
The brotherhood relation is symmetric: If x is a brother of y then clearly y is a brother of x because they both have the same biological parents.
The brotherhood relation is transitive: Suppose that x is a brother of y and y is a brother of z. Then x as the same two biological parents as y and y has the same two biological parents as z. Then x has the same two biological parents as z, hence x and z are brothers
We conclude that the brotherhood relation is an equivalence relation.

(b) The *descendent relation*: for two humans (dead or alive) we say that x is a descendent of y (or that y is an ancestor of x) is x is the son of a son of a son ... if a son of y. It is a matter of definition if this relation is reflexive, namely, is x is a descendent of himself. This is not symmetric, since for example, Jeffery Jordan is a descendent (the son of) Michael Jordan, but Michael Jordan is not the a descendent of Jeffery Jordan.[2]

(2) Let $A = \{1, 2, 3, 4, 5, 6\}$ then

$$E = \{\underbrace{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 4,4\rangle, \langle 5,5\rangle, \langle 6,6\rangle}_{id_A}, \langle 1,5\rangle, \langle 5,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle, \langle 3,6\rangle, \langle 6,3\rangle, \langle 2,6\rangle, \langle 6,2\rangle\}$$

is an equivalence relation on A.

(3) Among the most important equivalence relations is the congruence relation. Recall that for a natural number $n > 0$ and two integers z_1, z_2 we say that $z_1 \equiv z_2 \mod n$ if $z_1 \mod n = z_2 \mod n$. In order to avoid the use modulo in the definition congruency, we can formulate it as follows:

$$E_n = \{\langle z_1, z_2\rangle \in \mathbb{Z}^2 \mid z_1 - z_2 \text{ is divisible by } n\}$$

Let us prove that E_n is an equivalence relation.
Reflexive: we want to prove that for every $z \in \mathbb{Z}$, zE_nz. Let $z \in \mathbb{Z}$, we want to prove that $z - z = 0$ is divisible by n, but this is true sine every number divides 0(recall the formal definition of divisibility and

[1] This is simply a convenient choice of definition, one can consider other definitions for brotherhood.

[2] Note that in order to prove that a relation is not reflexive/symmetric/transitive we should always give a **specific** counter example, since these properties are universal properties and therefore their negation is an existential property.

prom this easy fact!).
Symmetric: We want to prove that for every $z_1, z_2 \in \mathbb{Z}$, if $z_1 E_n z_2$ then $z_2 E_n z_1$. Let $z_1, z_2 \in \mathbb{Z}$ and suppose (this is an implication!) that $z_1 E_n z_2$, we want to prove that $z_2 E_n z_1$. [3] By definition of E_n, we conclude that n divides $z_1 - z_2$ and therefore there is $k \in \mathbb{Z}$ such that $z_1 - z_2 = k \cdot n$. Hence $z_2 - z_1 = (-k) \cdot n$ and also $-k \in \mathbb{Z}$. It follows again by the definition of E_n that $z_2 E_n z_1$.
Transitive: Suppose that $z_1 E_n z_2$ and $z_2 E_n z_3$, we want to prove that $z_1 E_n z_3$. By definition of E_n, this means that n divides $z_1 - z_2$ and also $z_2 - z_3$. By definition f divisibility, there are $k_1, k_2 \in \mathbb{Z}$ such that $z_1 - z_2 = k_1 n$ and $z_2 - z_3 = k_2 n$. Summing the two equations, we get:

$$z_1 - z_3 = (z_1 - z_2) + (z_2 - z_3) = k_1 n + k_2 n = (k_1 + k_2)n)$$

Since $k_1 + k_2 \in \mathbb{Z}$, it follows that $z_1 - z_3$ is divisible by n. By the definition of E_n, it follow that $z_1 E_n z_3$.

We conclude that E_n is an equivalence relation.

(4) $S = \{\langle n, m\rangle \in \mathbb{Z}^2 \mid \exists k \in \mathbb{Z} n + k^2 = m\}$ is reflexive, not symmetric, since for example $0S1$ (as $0 + 1^2 = 1$) but $1 \not S 0$ (prove that!). It is not transitive since for example $1 + 1^2 = 2$ and $2 + 1^2 = 3$ however $3 - 1 = 2$ is not a square of a natural (or even rational) number.

(5) Let us prove that the relation

$$R = \left\{\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))^2 \mid ad = bc\right\}$$

we use to construct the rational numbers is indeed an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$:
Reflexive: Let $\langle a, b\rangle \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, [4] we want to prove that $\langle a, b\rangle R \langle a, b\rangle$. This follows, since $ab = ab$ and by the definition of R.
Symmetric: Suppose that $\langle a, b\rangle R \langle c, d\rangle$, we want to prove that $\langle c, d\rangle R \langle a, b\rangle$. By our assumption we see that $ad = bc$, and since we can switch the order of number multiplication we get that $da = cb$ and therefore $\langle c, d\rangle R \langle a, b\rangle$.
Transitive: Suppose that $\langle a, b\rangle R \langle c, d\rangle$, $\langle c, d\rangle R \langle e, f\rangle$. We want to prove that $\langle a, c\rangle R \langle e, f\rangle$. By the assumption we have that $ad = bc$ and $cf = de$. Note that $adf = bcf = bde$ and since[5] $d \neq 0$, we can eliminate it from the equation to see that $af = be$. By definition of R, it follows that $\langle a, b\rangle R \langle e, f\rangle$.

It follows that R is an equivalence relation.

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[3] Usually, we will start directly with "suppose that $z_1 E_n z_2$, we want to prove that $z_2 E_n z_1$".

[4] We want to prove that $\forall a \in A. aRa$. In our case $A = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is a set of pairs! hence we want to prove that $\forall \langle a, b\rangle \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}). \langle a, b\rangle R \langle a, b\rangle$.

[5] Indeed $\langle c, d\rangle \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$, $c \in \mathbb{Z}$ and $d \in \mathbb{Z} \setminus \{0\}$. Therefore $d \neq 0$.

(6) For any set A, the identity relation id_A and $A \times A$ are always equivalence relations on the set A.

(7) Here are two examples of equivalence relations on $\mathbb{R}^3$:
$$H_1 = \{\langle\langle a,b,c\rangle, \langle a',b',c'\rangle\rangle \in \mathbb{R}^3 \mid a = a'\}$$
$$H_2 = \{\langle\langle a,b,c\rangle, \langle a',b',c'\rangle\rangle \in \mathbb{R}^3 \mid a+b+c = a'+b'+c'\}.$$
The equivalence criterion that the relation H_1 sets is to identify between triples with the same first coordinate. The equivalence that H_2 sets is to identify triples with the same sum.

(8) Here is an equivalence relations on the set $P(\mathbb{N})\{\emptyset\}$:
$$T_1 = \{\langle X,Y\rangle \in (P(\mathbb{N}) \setminus \{\emptyset\})^2 \mid \min(X) = \min(Y)\}$$
T_1 identifies sets with the same minimal elements. Here is an equivalence relation on the set $P(\mathbb{N})$:
$$T_2 = \{\langle X,Y\rangle \in (P(\mathbb{N}) \setminus \{\emptyset\})^2 \mid X \cap \mathbb{N}_{even} = X \cap \mathbb{N}_{odd}\}$$
T_2 identifies sets which includes exactly the same even numbers.

Back to our example of the rational numbers, what is the object $\frac{1}{2}$? is it $\langle 1,2\rangle$ or is it $\langle 2,4\rangle$? the definition of $\frac{1}{2}$ is just the set of those pairs $\{\langle 1,2\rangle, \langle 2,4\rangle, \rangle 3,6\rangle, \langle -1,-2\rangle ....\}$. The point is that we "glue" together all the conditions which are equivalent to $\langle 1,2\rangle$. Formally, we call this an *equivalence class*:

Definition 2.7. Let E be an equivalence relation on a set A. The *equivalence class* of an element $a \in A$ is the set of all conditions $b \in A$ such that a is E-equivalent to b. Formally, we denote the equivalence class of a by
$$[a]_E = \{b \in A \mid aEb\}$$

Example 2.8. We use the same notations from the previous example.

(1) In the brotherhood relation we have for example the following equivalence classes:
$$[\text{Orville Wright}]_{brotherhood} = \{\text{Orville Wright, Wilbur Wright}\}$$
$$[\text{Steph Curry}]_{brotherhood} = \{\text{Steph Curry, Seth Curry, Sydel Curry}\}$$
$$[\text{Kim Kardashian}]_{brotherhood} = \{\text{Kim Kard., Kourtney Kard., Khloé Kard., Rob Kard.}\}$$

(2) For $A = \{1,2,3,4,5,6\}$ and E from example (2), We have that:
$$[1]_E = \{1,5\}$$
$$[2]_E = \{2,3,6\}$$
$$[3]_E = \{2,3,6\}$$
$$[4]_E = \{4\}$$
$$[5]_E = \{1,5\}$$
$$[6]_E = \{2,3,6\}$$
This is not a coincidence that $[1]_E = [5]_E$ and that $[2]_E = [3]_E = [6]_E$, can you guess way?

(3) The equivalence classes of E_n are
$$[0]_{E_n} = \{0, n, -n, 2n, -2n, 3n, ...\} = \{zn \mid z \in \mathbb{Z}\}$$
$$[1]_{E_n} = \{1, n-1, -n+1, 2n-1, -2n+1, ...\} = \{zn+1 \mid z \in \mathbb{Z}\}$$
A general equivalence class is just:
$$[i]_{E_n} = \{zn + i \mid z \in \mathbb{Z}\}$$
and $i \equiv j \mod n$ if and only if $[i]_{E_n} = [j]_{E_n}$.

(4) Using equivalence classes and the equivalence relation R we can now formally define the rational number $\frac{n}{m} = [\langle n, m\rangle]_R$. For example, the number $\frac{1}{2}$ is just $[\langle 1, 2\rangle]_R$. We will see later that $[\langle 1, 2\rangle]_R = [\langle 2, 4\rangle]_R$ for example, where the last equality is an actual set equality!

(5) The equivalence class of a general triple $\langle a, b, c\rangle \in \mathbb{R}^3$ has the form:
$$[\langle a, b, c\rangle]_{H_1} = \{\langle a, x, y\rangle \mid x, y \in \mathbb{R}\}$$
and
$$[\langle a, b, c\rangle]_{H_2} = \{\langle x, y, (a + b + c - x - y)\rangle \mid x, y \in \mathbb{R}\}$$

(6) We have fore example
$$[\{4, 7, 3, 22\}]_{T_1} = \{X \in P(\mathbb{N}) \mid 3 = \min(X)\}$$
and
$$[\{4, 7, 3, 22\}]_{T_2} = \{X \in P(\mathbb{N}) \mid X \cap \mathbb{N} = \{2, 22\}\}$$

Proposition 2.9. *Let E be an equivalence relation on A. Then for every $a, b \in A$:*

(1) Either $[a]_E = [b]_E$.

(2) Or $[a]_E \cap [b]_E = \emptyset$

Moreover, $[a]_E = [b]_E$ if and only if aEb.

Proof. Let $a, b \in A$. We formally need to prove a $\vee$-statement. Let us split into cases:

(1) Suppose $[a]_E \cap [b]_E = \emptyset$, the (2) holds and we are done.

(2) Suppose $[a]_E \cap [b]_E \neq \emptyset$. We want to prove that $[a]_E = [b]_E$, which is sets equality. Let us prove a double inclusion:

  (a) $\underline{[a]_E \subseteq [b]_E}$: Let $x \in [a]_E$. We want to prove that $x \in [b]_E$. Let $c \in [a]_E \cap [b]_E$, which exists by the assumption in this case. By definition of equivalence relation, xEa, cEa and cEb.
  - By symmetry, since cEa, then aEc.
  - By transitivity, since xEa and aEc, then xEc.
  - Again by trasitivity since xEc and cEb, xEb.

  By the definition of equivalence class it follows that $x \in [b]_E$.

  (b) $\underline{[b]_E \subseteq [a]_E}$: Follows from the symmetry between a and b.

This concludes the proof that $[a]_E = [b]_E$ or $[a]_R \cap [b]_E = \emptyset$. For the moreover part, we nee to prove a double implication:

(1) $\Longrightarrow$: Suppose that $[a]_E = [b]_E$, we need to prove that aEb. Since E is reflexive, aEa and therefore $a \in [a]_E$. By the equality of the set $[a]_E = [b]_E$ we conclude that $a \in [b]_E$ and by the definition of equivalence class we conclude that aEb.

(2) $\Longleftarrow$: Suppose that aEb, we need to prove that $[a]_E = [b]_E$. Again since E is reflexive we have that $a \in [a]_E$ and by the definition of equivalence class we have that $a \in [b]_E$. Thus $a \in [a]_E \cap [b]_E$, which means that $[a]_E \cap [b]_E \neq \emptyset$. By the first part, this must means that $[a]_E = [b]_E$.

□

Corollary 2.10. *The following are equivalent:*

(1) $a \not{E} b$.
(2) $[a]_E \neq [b]_E$.
(3) $[a]_E \cap [b]_E = \emptyset$.

Proof. exercise. □

Definition 2.11. Let E be an equivalence relation on A. The *quotient set* of A by E (a.k.a "A modulo E") is the set of **all** equivalence classes.[6]. We denote it by[7]

$$A/E = \{[a]_E \mid a \in A\}$$

Example 2.12. (1) The "set" Humans/brotherhood consist of all possible equivalence classes, each equivalence class is the set of siblings from a given family. We can label each equivalence class according to the family name and think of the quotient

Humans/brotherhood = {"The Kardeshians", "The Curry's", "The Wright's", ...}

(2) $A/E = \{\{1, 5\}, \{2, 3, 6\}, \{4\}\}$.

(3) We have that

$$\mathbb{Z}/E_n = \{\{zn + i \mid z \in \mathbb{Z}\} \mid i = 0, 1, 2, ..., n - 1\}$$

Since each equivalence class in E_n is associated with a residue modulo n, we think of $\mathbb{Z}/E_n$ as the sets of residues modulo n.

(4) The rational numbers are defined as

$$\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/R$$

(5)

$$\mathbb{R}^3/H_1 = \{\{\langle a, x, y\rangle \mid x, y \in \mathbb{R}\} \mid a \in \mathbb{R}\}$$

Here every equivalence class can be identified with a single real number a.

$$\mathbb{R}^3/H_2 = \{\{\langle x, y, (s - x - y)\rangle \mid x, y \in \mathbb{R}\} \mid s \in \mathbb{R}\}$$

Also here the equivalence classes can be identifies with a single real number s which represents the sum $a + b + c$.

[6]Needless to say, without repetitions.

[7]Do not confused A/E with set difference $A \setminus E$.

(6)

$$(P(\mathbb{N}) \setminus \{\emptyset\})/T_1 = \{\{X \in P(\mathbb{N}) \setminus \{\emptyset\} \mid \min(X) = n\} \mid n \in \mathbb{N}\}$$

And each equivalence class can be identified with a natural number.

$$P(\mathbb{N})/T_2 = \{\{X \in P(\mathbb{N}) \mid X \cap \mathbb{N}_{even} = Y\} \mid Y \in P(\mathbb{N}_{even})\}$$

And each equivalence class can be identified with a set of even numbers.